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JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 34, 602–617 (1971)

On the Structure of the Set of Solutions of Nonlinear Equations

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Submitted by Ky Fan

Let T be a mapping from a subset of a Banach space X into a Banach space Y. The present paper investigates the nature of the set of solutions of the equation $T(x) = y$ for a given $y \in Y$, i.e. when $T^{-1}(y) \neq \varnothing$? What are the topological properties of $T^{-1}(y)$? A prototype for an answer to these questions is given by Peano existence theorem on the connectedness of the set of solutions of an ordinary differential equation in the real case. In its general setting, this problem was first attacked by Aronszajn [1] and Stampacchia [11]; recently, by Browder-Gupta [5], Vidossich [12] and, above all, Browder [3, Sec. 5] who gives several interesting results in an excellent treatment. Customary, the structure of $T^{-1}(y)$ is studied under the assumption that T is a proper map (T is a *proper map* iff $T^{-1}(K)$ is compact whenever K is), since differential and integral equations in the finite dimensional case lead in a natural way to proper maps. But in the infinite dimensional case this correspondence holds rarely, so that one must find more suitable maps. Since the problem of what are the "good" mappings for the infinite dimensional case is still open, the results of the present paper may be of interest at least as first step toward the general solution. For, the investigation of $T^{-1}(y)$ is made here under the assumption that T is a closed map.

Section 1 gives a general existence theorem which is employed to deduce a proof of Peano existence theorem for ordinary differential equations so elementary that it may be included in any first course on differential equations.

Section 2 gives some theorems on the connectedness of $T^{-1}(y)$, generalizing in various ways theorems proved in [1, 3, 5 and 11], and solving positively in a special case the problem of Stampacchia treated in [12]. The elimination of the hypothesis "T is proper" in the present theorems is obtained by using in a non-trivial manner a theorem of Michael [10] recently generalized by Dykes [7], according to which a continuous closed onto map with suitable domain is compact-covering ($T : X \to Y$ is a *compact-covering map* iff for each compact $K \subseteq Y$ there is a compact $C \subseteq X$ such that $T(C) = K$. Note

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that $x \rightsquigarrow 0$ is a compact-covering map $\mathbf{R} \to \mathbf{R}$ which is not proper). A particular care has been made on the statements of these theorems in order to obtain their applicability to topological manifolds, since an interest for this may rise from [4].

Section 3 gives some application of the general results, by proving a fixed point theorem for nonexpansive mappings with unbounded domains and an existence theorem, with Peano phenomenon, for differential equations in Banach spaces. The fixed point theorem of nonexpansive mappings is used to show that some integral equations have Peano phenomenon, i.e. that their solutions form a connected set.

Notations. We shall denote by

$B(x, \epsilon)$	the closed ball centered at x with radius ϵ.
I	the identity map.
$\|\cdot\|_\infty$	the norm of uniform convergence.
$C(K, X)$	the Banach space of all continuous maps from the compact space K into the Banach space X, equipped with the norm $\|\cdot\|_\infty$.

1. Existence

In the most general setting, a nonlinear equation may be viewed as an equality

$$T(x) = y$$

with T a map $X \to Y$ and x the unknown. Therefore the corresponding existence problem is nothing else than

$$T^{-1}(y) \neq \varnothing.$$

In this section we prove a general, but easy, existence theorem. Its weaker version under the hypothesis "$\lim_n T_n = T$ uniformly on X" is used several times in [3] but never explicitly pointed out. Note that Condition (a) is fulfilled whenever $T(\bigcup_{n=1}^{\infty} T_n^{-1}(y))$ is closed in Y.

1.1 Theorem. *Let X be any set, Y any metric space, T and T_n ($n \in \mathbf{Z}^+$) arbitrary maps $X \to Y$. If for $y \in \bigcap_{n=1}^{\infty} T_n(X)$ the following conditions*

(a) *the closure of $T(\bigcup_{n=1}^{\infty} T_n^{-1}(y))$ in Y is contained in $T(X)$,*

(b) *$\lim_n T_n = T$ uniformly on $\bigcup_{n=1}^{\infty} T_n^{-1}(y)$,*

hold, then $T^{-1}(y) \neq \varnothing$.

Proof. For each n there is $x_n \in X$ such that $T_n(x_n) = y$. From

$$d(y, T(x_n)) = d(T_n(x_n), T(x_n))$$

and the uniform convergence of $(T_n)_n$ to T on

$$\{x_n \mid n \in \mathbf{Z}^+\} \subseteq \bigcup_{n=1}^{\infty} T_n^{-1}(y)$$

it follows $\lim_n T(x_n) = y$. Then y belongs to the closed set

$$\overline{T\left(\bigcup_{n=1}^{\infty} T_n^{-1}(y)\right)} \subseteq T(X).$$

Q.E.D.

We note that 1.1 gives a proof of Peano existence theorem (as well as of some well known existence theorems in case of Banach spaces) for ordinary differential equations so easy and elementary that it can be included in a first course on differential equations, primarily since it avoids any use of fixed point theorems. For, let $f : [0, 1] \times \mathbf{R}^n \to \mathbf{R}^n$ be a mapping which is continuous (or, more generally, satisfies Carathéodory hypotheses) and fulfills the following condition

$$\|f(t, x)\| \leqslant g(t) \qquad (\text{all } t, x)$$

with $g : [0, 1] \to \mathbf{R}$ summable. Put

$$T(x)(t) = x(t) - \int_0^t f(\xi, x(\xi))\, d\xi,$$

$$T_n(x)(t) = \begin{cases} x(t) & \text{for } t \leqslant \dfrac{1}{n}, \\ x(t) - \displaystyle\int_0^{t-1/n} f(\xi, x(\xi))\, d\xi & \text{for } t \geqslant \dfrac{1}{n}. \end{cases}$$

the T_n's being known as Tonnelli's approximants. By examining each interval $[0, 1/n]$, $[1/n, 2/n]$,..., one can easily verifies that each

$$T_n : C([0, 1], \mathbf{R}^n) \to C([0, 1], \mathbf{R}^n)$$

is onto. The closedness of $T(C([0, 1], \mathbf{R}^n))$ follows from Ascoli theorem, while $\lim_n T_n = T$ uniformly follows from the uniform continuity of $t \rightsquigarrow \int_0^t g(\xi)\, d\xi$. Then T is onto by 1.1, what means that the differential equation

$$x' = y + f(t, x)$$

has some solution for every $y \in C([0, 1], \mathbf{R}^n)$.

2. P-Uniqueness

Peano proved that the set of integral curves of a differential equation in the real case has the following property: it meets each line parallel to the y-axis in a connected set. Successively, Aronszajn [1], Stampacchia [11] and Browder-Gupta [5] proved a general theorem which implies that the set of integral curves is a connected subset of the space of continuous maps, a result stronger than that of Peano. We label

P-uniqueness

the property that the set of solutions is a connected set, P-uniqueness standing for "Peano uniqueness" or "pseudo-uniqueness".

The aim of this section is a study of P-uniqueness in its most general setting. In the author's opinion, this section is a suite to [3, Sec. 5]. In analogy to [3, Sec. 5] and to uniqueness theorems for ordinary differential equations, we shall always assume the existence of solutions—whose study has been made in Sec. 1.

Recall that a map $T : X \to Y$ is said to be *open* (resp.: *closed*) iff $T(A)$ is an open (resp.: a closed) subset of Y whenever A is of X.

2.1 Lemma. *Let X, Y be topological spaces, and $T : X \to Y$ an open onto map such that $T^{-1}(y)$ is connected for all $y \in Y$. Then $T^{-1}(C)$ is connected whenever C is.*

Proof. If $T^{-1}(C)$ is not connected, then there are two nonempty disjoint open subsets U', U'' of $T^{-1}(C)$ whose union is $T^{-1}(C)$. Let V', V'' be open subsets of X such that $V' \cap T^{-1}(C) = U'$ and $V'' \cap T^{-1}(C) = U''$. Then $T(V')$ and $T(V'')$ are nonempty open subsets of Y whose union contains C. Therefore, by the connectedness of C, there is

$$y \in (C \cap T(V')) \cap (C \cap T(V'')).$$

If $x \in V' \cap T^{-1}(y)$ (resp.: $x \in V'' \cap T^{-1}(y)$), then $x \in T^{-1}(C)$, so that $x \in V' \cap T^{-1}(C) = U'$ (resp.: $x \in V'' \cap T^{-1}(C) = U''$). Consequently $U' \cap T^{-1}(y)$ and $U'' \cap T^{-1}(y)$ are nonempty disjoint open subsets of $T^{-1}(y)$ whose union is $T^{-1}(y)$. But this is impossible since $T^{-1}(y)$ is connected by hypotheses. Q.E.D.

The above lemma includes a well known result on quotients of topological spaces under open equivalent relations.

In order to simplify the statements of the following theorems, we introduce the following

DEFINITION. A map $T : X \to Y$, X and Y being topological spaces, will be called *quasi-invertible* iff $T^{-1}(y)$ is connected for each $y \in Y$ and T sends open subsets of X into open subsets of $T(X)$.

According to [3, Sec. 5], a map $T : X \to Y$ is called *semi-invertible* whenever $T^{-1}(y)$ is connected for all $y \in Y$ and T is a closed map (what is equivalent to: $y \rightsquigarrow T^{-1}(y)$ is an upper semi-continuous map $Y \to 2^X$). By [3, Proposition 5.1] and 2.1, semi-invertible and quasi-invertible maps both have the property that inverse images by them of connected sets in the range are connected. Looking only at this property, the sole of our interest, in defining semi-invertible maps it is enough to assume that all $T^{-1}(y)$ are connected and T is a closed map $X \to T(X)$. For, we may repeat the argument in the proof of 2.1 for closed sets (instead of open sets), to show that $T^{-1}(C)$ is connected whenever $C \subseteq T(X)$ is, T being semi-invertible in this new sense.

The following theorem generalizes [3, Theorem 5.8] in various ways.

2.2 THEOREM. *Let X be a paracompact Hausdorff space, and Y a metric space. Let $T : X \to Y$ be a continuous map sending closed subsets of X into closed subsets of $T(X)$. For each $n \in \mathbf{Z}^+$, let T_n be a semi-invertible or a quasi-invertible map $X \to Y$. If $y \in T(X)$ is an interior point of $\bigcap_{n=1}^{\infty} T_n(X)$ with a base of connected neighborhoods, then $T^{-1}(y)$ is connected whenever* $\lim_n T_n = T$ *uniformly on $T^{-1}(y)$ and on all sets of the form $\bigcup_{n=1}^{\infty} T_n^{-1}(C)$ with C a compact subset of $\bigcap_{n=1}^{\infty} T_n(X)$.*

By using [7, Theorem 3.4] instead of [10, 1.2], the proof of 2.2 carries over verbatim in the more general case X is normal and admits a complete uniformity. The hypothesis "y is an interior point of $\bigcap_{n=1}^{\infty} T_n(X)$" cannot longer be relaxed as shown by [12]. Finally, note that 1.1 implies that $y \in T(X)$ whenever T is a closed map $X \to Y$.

Examine now some example of closed maps. Any homeomorphism into $T : X \to Y$ is a closed map $X \to T(X)$, while any proper map $X \to Y$ is a closed map $X \to Y$. Consequently $I - f$ is a closed map $K \to X$ whenever $f : K \subseteq X \to X$ is compact or condensive (condensive maps are the generalization of the "α-contrazioni" of Darbo [6] given by [3]) and K a closed subset of the Banach space X. Let X be a Banach space, $K \subseteq X$ closed and $f : K \to X$ such that

$$c \| x - y \| \leqslant \| f(x) - f(y) \|, \qquad (x, y \in K)$$

with c a constant >1. Then $T = I - f$ is a closed map $X \to T(X)$, as it follows from the following inequalities:

$$c \| x_n - x \| \leqslant \| f(x_n) - f(x) \| \leqslant \| x_n - x \| + \| T(x_n) - T(x) \|$$
$$(c - 1) \| x_n - x \| \leqslant \| T(x_n) - T(x) \| .$$

Proof of 2.2. Assume $F = T^{-1}(y)$ disconnected, and argue for a contradiction. There are a sequence $(U_n)_n$ of connected neighborhoods of y in Y and a sequence $(\delta_n)_n$ of positive real numbers tending to 0 such that

$$U_{n+1} \subseteq B(y, \delta_n) \subseteq U_n \subseteq \bigcap_{k=1}^{\infty} T_k(X), \qquad (n \in \mathbf{Z}^+). \tag{1}$$

By passing to a subsequence if necessary, we assume

$$d(T(x), T_n(x)) \leqslant \delta_n \qquad (n \in \mathbf{Z}^+;\, x \in F) \tag{2}$$

where d is the metric of Y. The sets $F_n = T_n^{-1}(U_n)$ $(n \in \mathbf{Z}^+)$ are connected by 2.1 and [3, Proposition 5.1]. From (1) it follows

$$d(y, T_{n+1}(x)) \leqslant \delta_n \qquad (n \in \mathbf{Z}^+;\, x \in F_{n+1}). \tag{3}$$

Since F is disconnected, there are two nonempty disjoint closed subsets F', F'' of F with F as union. Since X is a normal space, there are two disjoint open sets U', U'' of X such that $F' \subseteq U'$ and $F'' \subseteq U''$. By (2) and (1), $F \subseteq F_n$ for all n, so that

$$U' \cap F_n \neq \varnothing \neq U'' \cap F_n \qquad (n \in \mathbf{Z}^+).$$

Therefore, by the connectedness of F_n, we shall obtain the wanted contradiction by showing the existence of $n \in \mathbf{Z}^+$ such that $F_n \subseteq U$, where $U = U' \cup U''$. Assume that for all n there is $x_n \in F_n \backslash U$, and argue again for a contradiction. Let C be the compact subset $\{y\} \cup \{T_n(x_n) \mid n \in \mathbf{Z}^+\}$ of $\bigcap_{n=1}^{\infty} T_n(X)$, the compactness being a consequence of (3). From

$$d(y, T(x_n)) \leqslant d(y, T_n(x_n)) + d(T_n(x_n), T(x_n)) \qquad (n \in \mathbf{Z}^+), \tag{4}$$

by (3) and the uniform convergence of $(T_n)_n$ to T on

$$\{x_n \mid n \in \mathbf{Z}^+\} \subseteq \bigcup_{n=1}^{\infty} T_n^{-1}(C),$$

it follows $\lim_n T(x_n) = y$. Two cases are possible:

Case 1. $\{T(x_n) \mid n \in \mathbf{Z}^+\}$ is a finite set. Then $\lim_n T(x_n) = y$ implies the existence of $n \in \mathbf{Z}^+$ such that $T(x_n) = y$, so that $x_n \in F \backslash U$, a contradiction.

Case 2. $\{T(x_n) \mid n \in \mathbf{Z}^+\}$ is infinite. Put $F_0 = F \cup \{x_n \mid n \in \mathbf{Z}^+\}$. Let $\mathfrak{u}$ be an ultrafilter in $\{x_n \mid n \in \mathbf{Z}^+\}$ converging to a point x_0 of X. If $\mathfrak{u}$ contains a finite set A, then $x_0 \in A \subseteq \{x_n \mid n \in \mathbf{Z}^+\}$. If all $A \in \mathfrak{u}$ are infinite, then $\mathfrak{u}$ is finer than the filter with base $\{\{x_n \mid n \geqslant m\} \mid m \in \mathbf{Z}^+\}$, so that $T(\mathfrak{u})$ is finer than the filter with base $\{\{T(x_n) \mid n \geqslant m\} \mid m \in \mathbf{Z}^+\}$. Then

$$T(x_0) = \lim_{\mathfrak{u}} T = \lim_n T(x_n) = y,$$

and consequently $x_0 \in F$. This shows that

$$F_0 = F \cup \overline{\{x_n \mid n \in \mathbf{Z}^+\}},$$

so that F_0 is a closed subset of X. Consequently $T|_{F_0}$ is a closed map $F_0 \to T(F_0) \subseteq T(X)$, and F_0 a paracompact space. Then [10, Corollary 1.2] implies that $T|_{F_0}$ is a compact-covering map $F_0 \to T(F_0)$. Since

$$\lim_n T(x_n) = y, \qquad T(F_0) = \{y\} \cup \{T(x_n) \mid n \in \mathbf{Z}^+\}$$

is a compact set. Therefore there is a compact $K \subseteq F_0$ such that $T(K) = T(F_0)$. Since $T(F_0)\backslash\{y\}$ is infinite, $J = \{n \in \mathbf{Z}^+ \mid x_n \in K\backslash F\}$ must be infinite. Then $(x_n)_{n \in J}$ is a subsequence of $(x_n)_{n=1}^{\infty}$. Since $(x_n)_{n \in J}$ is a sequence in the compact space K, there is a subnet $(z_\alpha)_\alpha$ of $(x_n)_{n \in J}$ converging to a point $x \in K$. From $T(x) = \lim_\alpha T(z_\alpha) = \lim_n T(x_n) = y$ it follows $x \in F \subseteq U$. But this contradicts the fact that $x \in X\backslash U$ since $(z_\alpha)_\alpha$ is a net in the closed set $X\backslash U$.

Q.E.D.

The following corollary gives in a particular case an affirmative answer to G. Stampacchia's problem concerning the closedness of the set of continuous proper maps with Peano phenomenon (see [12], where it is shown that this problem cannot have a general affirmative answer). For, it implies that in the set of continuous proper maps $A \to K$, where A is an arbitrary subset of a Banach space and K a closed convex subset of a Banach space, the subset of onto maps possessing Peano phenomenon is closed for the topology of uniform convergence.

2.3 COROLLARY. *Let X be a paracompact Hausdorff space and Y a locally connected metric space. In the set of semi-invertible maps $X \to Y$ with closed range, the subset of continuous onto maps is closed for the topology of uniform convergence on X.*

Proof. By 1.1 and 2.2. Q.E.D.

The following theorem generalizes Aronszajn-Stampacchia-Browder-Gupta theorem [1], [11], [5, Theorem 7], [3, Theorem 5.9]. Note that y has a neighborhood homeomorphic to a closed convex subset of a Fréchet space whenever it has a neighborhood homeomorphic to an open convex subset of a Fréchet space, so that the following theorem applies to topological manifolds modelled on Banach or Fréchet spaces. Recall that an R_δ is the intersection of a decreasing sequence of compact absolute retracts (hence an R_δ is connected).

2.4 THEOREM. *Let X be a metric space, Y a metric space and y a point of Y with a neighborhood homeomorphic to a closed convex subset of a Fréchet space. Let T be a continuous map $X \to Y$ which is a closed map $X \to T(X)$, and $T_n : X \to Y$ ($n \in \mathbf{Z}^+$) a homeomorphism into. If y is an interior point of $\bigcap_{n=1}^{\infty} T_n(X)$ and $T^{-1}(y)$ compact and nonempty, then $T^{-1}(y)$ is an R_δ whenever $\lim_n T_n = T$ uniformly on $T^{-1}(y)$ and on all set of the form $\bigcup_{n=1}^{\infty} T_n^{-1}(C)$ with C a compact subset of $\bigcap_{n=1}^{\infty} T_n(X)$.*

Proof. There are a sequence $(U_n)_n$ of neighborhoods of y in Y and a sequence $(\delta_n)_n$ of positive real numbers tending to 0 such that

$$U_{n+1} \subseteq B(y, \delta_n) \subseteq U_n \subseteq \bigcap_{m=1}^{\infty} T_m(X) \qquad (n \in \mathbf{Z}^+) \tag{1}$$

and, for each n, U_n is homeomorphic, say by φ_n, to a closed convex subset K_n of a Fréchet space. By passing to a subsequence if necessary, we assume

$$d(T(x), T_n(x)) \leqslant \delta_n \qquad (n \in \mathbf{Z}^+;\, x \in F) \tag{2}$$

where d is the metric of Y and $F = T^{-1}(y)$. By (1) and (2), $T_n(F) \subseteq U_n$, so that the closed convex hull C_n of $\varphi_n(T_n(F))$ is a subset of K_n. By Mazur theorem, C_n is compact, while by Dugundji extension theorem C_n is AE (metric). Consequently

$$F_n = (\varphi_n \circ T_n)^{-1}\,(C_n) \qquad (n \in \mathbf{Z}^+)$$

are compact AE (metric). From (1) it follows

$$d(y, T_{n+1}(x)) \leqslant \delta_n \qquad (n \in \mathbf{Z}^+;\, x \in F_{n+1}). \tag{3}$$

From now on we may repeat the argument of Cases 1 and 2 in the proof of 2.2 in order to show that each neighborhood of F contains a subsequence of $(F_n)_n$. But then F is an R_δ by [3, Proposition 5.2] (or [5, Lemma 5], since it is easy to see that $F = \lim F_n$). Q.E.D.

It would be better to prove the above theorems under the assumption "$\lim_n T_n = T$ uniformly on compacta". The following theorem is an unsatisfactory result of this type, the unsatisfactoriness being due to the algebraic assumption concerning the convex inverse images by T_n's.

2.5 THEOREM. *Let X, Y be Fréchet spaces, $K \subseteq X$ closed, $T : K \to Y$ continuous and T_n ($n \in \mathbf{Z}^+$) arbitrary maps $K \to Y$ such that $T_n^{-1}(C)$ is convex whenever C is. If* $\lim_n T_n = T$ *uniformly on finite-dimensional compact subsets of K, then $T^{-1}(y)$ is connected for every interior point y of $\bigcap_{n=1}^{\infty} T_n(K)$.*

Proof. Fix $x_0 \in T^{-1}(y)$. Let $(X_\alpha)_\alpha$ be the family of all finite-dimensional vector subspaces of X containing x_0. For each α and each $n \in \mathbf{Z}^+$, let B_n^α be the closed ball of X_α with center x_0 and radius n. There are a sequence $(U_m)_m$ of convex neighborhoods of y and a sequence $(\delta_m)_m$ of positive real numbers tending to 0 such that, by fixing a metric d for Y, we have

$$U_{m+1} \subseteq B(y, \delta_m) \subseteq U_m \subseteq \bigcap_{k=1}^{\infty} T_k(K) \qquad (m \in \mathbf{Z}^+). \tag{1}$$

Fix α and n for the moment. Since $K \cap B_n^\alpha$ is a finite-dimensional compact subset of X, we may assume, by passing to a subsequence if necessary, that

$$d(T(x), T_m(x)) \leqslant \delta_m \qquad (x \in K \cap B_n^\alpha;\ m \in \mathbf{Z}^+). \tag{2}$$

Put $F_n^\alpha = T^{-1}(y) \cap B_n^\alpha$ and $F_m = T_m^{-1}(U_m) \cap B_n^\alpha$. Since $T_m^{-1}(U_m)$ is convex, F_m is convex. By (2) and (1), $F_n^\alpha \subseteq F_m$ for every m, so that

$$F_n^\alpha \subseteq \liminf F_m \subseteq \limsup F_m .$$

If $x \in \limsup F_m$, then there are a subsequence $(F_{m_k})_k$ of $(F_m)_m$ and $x_{m_k} \in F_{m_k}$ ($k \in \mathbf{Z}^+$) such that $\lim_k x_{m_k} = x$. By the continuity of T it follows $\lim_k T(x_{m_k}) = T(x)$. By (1) and (2), from

$$d(y, T(x_{m_k})) \leqslant d(y, T_{m_k}(x_{m_k})) + d(T_{m_k}(x_{m_k}), T(x_{m_k}))$$

it follows $\lim_k T(x_{m_k}) = y$. Then $y = T(x)$, and $x \in F_n^\alpha$. Thus $F_n^\alpha = \lim F_m$. Consequently F_n^α is connected by [9, Theorem 2-101]. Since

$$T^{-1}(y) = \bigcup_{\alpha, n} F_n^\alpha,$$

$T^{-1}(y)$ is connected by the well known fact that the union of a family of connected sets is connected whenever its intersection is not empty. Q.E.D.

3. APPLICATIONS

The aim of this section is to derive some existence and P-uniqueness theorem from the previous general results.

The following theorem generalizes [12, 2.1]. Condition (a) cannot be weakened to

$$\inf_{\epsilon>0} \sup_{\|x\|\geqslant\epsilon} \frac{\|f(x)\|}{\|x\|} \leqslant 1$$

as shown be the trivial example $f : x \rightsquigarrow x + x_0$ with $x_0 \neq 0$. Recall that f is called *nonexpansive* iff

$$\|f(x) - f(y)\| \leqslant \|x - y\| \qquad (\text{all } x, y).$$

3.1 THEOREM. *Let X be a Banach space and $f : X \to X$ nonexpansive. If $I - f$ sends bounded closed sets into closed sets, and if at least one of the following conditions*

(a) $\inf_{\epsilon>0} \sup_{\|x\|\geqslant\epsilon} \dfrac{\|f(x)\|}{\|x\|} < 1,$

(b) *f is a contraction for points sufficiently far, i.e. there are $\epsilon > 0$ and $k \in [0, 1[$ such that $\|f(x) - f(y)\| \leqslant k\|x - y\|$ whenever $\|x\| \geqslant \epsilon$,*

(c) *$f(X)$ is bounded,*

holds, then the set of fixed points of f is a nonempty connected set.

Proof. Clearly (c) implies (a), while (b) implies (a) since

$$\|f(x) - f(0)\| \leqslant k\|x\|.$$

Therefore it is enough to prove the theorem assuming (a). Then there are $\epsilon' > 0$ and $\delta \in [0, 1[$ such that

$$\sup_{\|x\|\geqslant\epsilon'} \frac{\|f(x)\|}{\|x\|} \leqslant \delta. \tag{1}$$

Since f is nonexpansive, $f(B(0, \epsilon'))$ is bounded. Fix $\delta_0 > \operatorname{diam} f(B(0, \epsilon'))$, $\epsilon_0 \geqslant \max(\epsilon', 2\delta_0)$, and $\alpha > 0$ such that

$$\alpha \leqslant \min\left(\frac{\delta_0}{2}, \frac{\epsilon'(1-\delta)}{2}\right). \tag{2}$$

Choose $\epsilon'' \geqslant \epsilon_0$. Put

$$g = f|_{B(0,\epsilon'')}, \qquad g_n = \left(1 - \frac{1}{n}\right)g, \qquad T_n = I - g_n, \qquad \text{and} \qquad T = I - g.$$

Since $g(B(0, \epsilon''))$ is bounded by the nonexpansivity of g, $\lim_n g_n = g$ and $\lim_n T_n = T$ uniformly on $B(0, \epsilon'')$. Therefore there is $n_0 \in \mathbf{Z}^+$ such that

$$\| g(x) - g_n(x)\| \leqslant \min\left(\frac{\delta_0}{2}, \frac{\epsilon'(1-\delta)}{2}\right) \qquad (x \in B(0, \epsilon''); n \geqslant n_0). \tag{3}$$

For each $n \geqslant n_0$ and each $y \in B(0, \alpha)$, if $\epsilon' \leqslant \| x \| \leqslant \epsilon''$ then

$$\begin{aligned}
\frac{\| g_n(x) + y \|}{\| x \|} &\leqslant \frac{\| g_n(x)\|}{\| x \|} + \frac{\| y \|}{\| x \|} \\
&\leqslant \frac{\| g_n(x) - g(x)\|}{\| x \|} + \frac{\| g(x)\|}{\| x \|} + \frac{\| y \|}{\| x \|} \\
&\leqslant \frac{\epsilon'(1-\delta)/2}{\epsilon'} + \delta + \frac{\epsilon'(1-\delta)/2}{\epsilon'} \qquad \text{(by (2), (1), (3))} \\
&\leqslant 1,
\end{aligned}$$

while if $\| x \| \leqslant \epsilon'$ then

$$\begin{aligned}
\| g_n(x) + y \| &\leqslant \| g_n(x) - g(x)\| + \| g(x)\| + \| y \| \\
&\leqslant \frac{\delta_0}{2} + \delta_0 + \frac{\delta_0}{2} \qquad \text{(by (3), (1))} \\
&= 2\delta_0 \leqslant \epsilon_0 \leqslant \epsilon''.
\end{aligned}$$

Therefore $x \rightsquigarrow g_n(x) + y$ maps $B(0, \epsilon'')$ into $B(0, \epsilon'')$, so that this mapping has a fixed point by contraction theorem ($n \geqslant n_0$; $\| y \| \leqslant \alpha$). Consequently

$$B(0, \alpha) \subseteq \bigcap_{n \geqslant n_0} T_n(B(0, \epsilon'')).$$

But then, T being a closed map $B(0, \epsilon'') \to X$ (so that $T(B(0, \epsilon''))$ is a closed subset of X) by the hypotheses, $0 \in T(B(0, \epsilon''))$ by 1.1, while by 2.2 $F_{\epsilon''} = T^{-1}(0)$ is connected. Since

$$(I - f)^{-1}(0) = \bigcup_{\epsilon'' \geqslant \epsilon_0} F_{\epsilon''}$$

and since $\bigcap_{\epsilon'' \geqslant \epsilon_0} F_{\epsilon''} \supseteq F_{\epsilon_0} \neq \varnothing$, $(I - f)^{-1}(0)$ is connected because the union of a family of connected sets is connected whenever its intersection is not empty. Q.E.D.

3.2 *Remark.* Let $f : X \to X$ be a nonexpansive or a condensive map. We may repeat the above argument to find an $\epsilon_0 > 0$ such that

$f(B(0, \epsilon_0)) \subseteq B(0, \epsilon_0)$. Therefore conditions (a),..., (c) of 3.1 are sufficient to guarantee the existence of a fixed point of f whenever f is condensive, as well as whenever f is nonexpansive and suitable hypotheses are assumed on X or f in order that $f|_{B(0,\epsilon_0)}$ has a fixed point (see [2] and [3]).

The following corollary cannot longer be improved as shown by the case $f = I$.

3.3 Corollary. *Let X be a Banach space and f a nonexpansive map $X \to X$ satisfying at least one of the conditions* (a),..., (c) *of* 3.1. *Then the vector field $I - f : X \to X$ is semi-invertible onto iff it is a closed map.*

Proof. A semi-invertible onto map is a closed map. Conversely, $(I - f)^{-1}(y)$ is nothing else than the set of fixed points of $x \rightsquigarrow f(x) + y$, so that it is enough to apply 3.1 (note that $x \rightsquigarrow x - f(x) - y$ is a closed map being the composition of $I - f$ and the homeomorphism $x \rightsquigarrow x - y$).

Q.E.D.

The sole hypothesis

$$\|f(t, s, x) - f(t, s, y)\| \leqslant \|x - y\| \qquad (\text{all } t, s, x, y)$$

is unsufficient to guarantee the existence of solutions of the integral equation

$$x(t) = y(t) + \int_0^1 f(t, \xi, x(\xi))\, d\xi$$

for every $y \in C([0, 1], \mathbf{R}^n)$. For, the integral equation

$$x(t) = y(t) + \int_0^1 x(\xi)\, d\xi$$

may be solved only if $\int_0^1 y(\xi)\, d\xi = 0$.

3.4 Corollary. *Let $f : [0, 1] \times [0, 1] \times \mathbf{R}^n \to \mathbf{R}^n$ be continuous such that*

$$\|f(t, s, x) - f(t, s, y)\| \leqslant \|x - y\| \qquad (\textit{all } t, s, x, y).$$

for any fixed norm $\|\cdot\|$ on $\mathbf{R}^n$. If the range of f is bounded, or if there are $\epsilon > 0$ and $1 > k \geqslant 0$ such that

$$\|f(t, s, x) - f(t, s, 0)\| \leqslant k\|x\| \qquad (0 \leqslant t, s \leqslant 1; \|x\| \geqslant \epsilon),$$

then the solutions of the integral equation

$$x(t) = y(t) + \int_0^1 f(t, \xi, x(\xi))\, d\xi$$

form a nonempty closed connected subset of $C([0, 1], \mathbf{R}^n)$.

Proof. Define $F : C([0, 1], \mathbf{R}^n) \to C([0, 1], \mathbf{R}^n)$ by

$$F(x)(t) = y(t) + \int_0^1 f(t, \xi, x(\xi))\, d\xi.$$

Since f is uniformly continuous on compact subsets of $[0, 1]^2 \times \mathbf{R}^n$, F sends bounded subsets into relatively compact sets. Therefore $I - F$ sends bounded closed sets into closed sets. Clearly F is nonexpansive. If f is bounded, then F is also, so that the conclusion follows from 3.1(c). Otherwise, there are $\epsilon > 0$ and $0 \leqslant k < 1$ such that

$$\| f(t, s, x) - f(t, s, 0)\| \leqslant k \| x \| \qquad (0 \leqslant t, s \leqslant 1; \| x \| \geqslant \epsilon).$$

Choose $x \in C([0, 1], \mathbf{R}^n)$ such that $\| x \|_\infty \geqslant 2\epsilon$. Then

$$\| F(x)\|_\infty \leqslant \sup_{0 \leqslant t \leqslant 1} \left(\| y \|_\infty + \int_0^1 \| f(t, \xi, x(\xi))\| \, d\xi\right).$$

From

$$f(t, \xi, x(\xi)) \leqslant \begin{cases} k \| x(\xi)\| + \| f(t, \xi, 0)\| & \text{if} \quad \| x(\xi)\| \geqslant \epsilon, \\ \| x(\xi)\| + \| f(t, \xi, 0)\| & \text{if} \quad \| x(\xi)\| < \epsilon, \end{cases}$$

it follows

$$\| F(x)\|_\infty \leqslant \max(\| y \|_\infty + k \| x \|_\infty + M, \| y \|_\infty + \epsilon + M)$$

where

$$M = \max_{t,s} \| f(t, s, 0)\|.$$

Therefore for each $n \geqslant 2$ we have

$$\sup_{\|x\|_\infty \geqslant n\epsilon} \frac{\| F(x)\|_\infty}{\| x \|_\infty} \leqslant \max \left(\frac{\| y \|_\infty}{n\epsilon} + k + \frac{M}{n\epsilon}, \frac{\| y \|_\infty}{n\epsilon} + \frac{1}{n} + \frac{M}{n\epsilon}\right).$$

Hence

$$\sup_{\|x\|_\infty \geqslant n\epsilon} \frac{\| F(x)\|_\infty}{\| x \|_\infty} < 1$$

for n sufficiently large, so that the conclusion follows from 3.1(a). Q.E.D.

Recall that the *Carathéodory hypotheses* on $f : [0, b] \times X \to X$, X being a Banach space, are:

(C_1) *$f(\cdot, x)$ is measurable for every $x \in X$.*

(C_2) *$f(t, \cdot)$ is continuous for almost every $t \in [0, b]$.*

(C_3) *$\| f(t, x)\| \leqslant g(t)$ with $g : [0, b] \to \mathbf{R}^+$ summable.*

In the following statement the word "countable" is unnecessary whenever X is separable, in view of the continuity of $f(t, \cdot)$. Hypothesis (C_3) may be omitted if there are two constants $A, B \geqslant 0$ such that

$$\| f(t, x)\| \leqslant A \| x \| + B \qquad (\text{all } t, x).$$

Recall that a subset of a Banach space is called *boundedly compact* iff all its bounded subsets are relatively compact. Hence every subset of a finite-dimensional Banach space is boundedly compact.

3.5 THEOREM. *Let X be a Banach space, $f : [0, b] \times X \to X$ with $f = f_1 + f_2$ such that for each countable bounded subset C of X, there are two constants A_C, $B_C \geqslant 0$ and a summable $u_C : [0, b] \to \mathbf{R}^+$ such that*

(1) $\| f_1(t, x)\| \leqslant A_C \| x \| + B_C, \qquad (0 \leqslant t \leqslant b; x \in C);$

(2) $\| f_2(t, x) - f_2(t, y)\| \leqslant u_C(t) \| x - y \|, \qquad (0 \leqslant t \leqslant b; x, y \in C).$

If f_1 is uniformly continuous on bounded subsets and

$$\{ f_1(t, x) \mid x \in X\}$$

is boundedly compact for every $t \in [0, b]$, if f_2 and f satisfy Carathéodory hypotheses, then the set of solutions of the generalized Cauchy problem

$$x(t) = y(t) + \int_0^t f(\xi, x(\xi)) \, d\xi$$

is an R_δ of $C([0, b], X)$ for every $y \in C([0, b], X)$.

Proof. Put $Y = C([0, b], X)$. Define $T : Y \to Y$ by

$$T(x)(t) = x(t) - \int_0^t f(\xi, x(\xi)) \, d\xi.$$

Let us show that T is proper, i.e. that if $(T(x_n))_{n=1}^{\infty}$ is a convergent sequence, then there is a convergent subsequence of $(x_n)_n$. By Carathéodory hypothesis (C_3) and the convergence of $(T(x_n))_n$, $\{x_n \mid n \in \mathbf{Z}^+\}$ is a bounded set. Let E be a countable dense subset of $[0, b]$. To the countable bounded set

$$C = \{x_n(t) \mid t \in E, n \in \mathbf{Z}^+\}$$

there correspond A_C, B_C and u_C such that (1) and (2) hold. From (1) and (2), by the continuity of $f_1(t, \cdot)$ and $f_2(t, \cdot)$, it follows

(i) $\| f_1(t, x_n(t))\| \leqslant A_C \| x_n(t)\| + B_C \qquad (0 \leqslant t \leqslant b; n \in \mathbf{Z}^+);$

(ii) $\| f_2(t, x_n(t)) - f_2(t, x_m(t))\| \leqslant u_C(t) \| x_n(t) - x_m(t)\|$
(almost every $0 \leqslant t \leqslant b; n, m \in \mathbf{Z}^+$).

By (i), $\{f_1(t, x_n(t)) \mid n \in \mathbf{Z}^+\}$ is bounded for all t, hence relatively compact. By the equicontinuity of $\{T(x_n) \mid n \in \mathbf{Z}^+\}$ and by (C_3), $\{x_n \mid n \in \mathbf{Z}^+\}$ is equicontinuous. Then

$$H = \{f_1(\cdot, x_n(\cdot)) \mid n \in \mathbf{Z}^+\}$$

is equicontinuous by the uniform continuity of f_1 on $\{x_n(t) \mid 0 \leqslant t \leqslant b, n \in \mathbf{Z}^+\}$. Therefore H is relatively compact by Ascoli theorem. Since uniform convergence of derivatives implies uniform convergence of mappings, there is a subsequence $(x_{n_k})_k$ of $(x_n)_n$ such that $(\int_0^t f_2(\xi, x_{n_k}(\xi))\, d\xi)_{k=1}^{\infty}$ is convergent in Y. Then for every $\epsilon > 0$ there is $k_0 \in \mathbf{Z}^+$ such that

$$\| T(x_{n_k}) - T(x_{n_j})\|_\infty \leqslant \epsilon$$

and

$$\sup_t \left\| \int_0^t (f_1(\xi, x_{n_k}(\xi)) - f_1(\xi, x_{n_j}(\xi)))\, d\xi \right\| \leqslant \epsilon$$

for $k, j \geqslant k_0$. Hence for $k, j \geqslant k_0$ we have

$$\begin{aligned}
\| x_{n_k}(t) - x_{n_j}(t)\| &\leqslant \| T(x_{n_k})(t) - T(x_{n_j})(t)\| + \left\| \int_0^t f_1(\xi, x_{n_k}(\xi)) \right. \\
&\qquad \left. - f_1(\xi, x_{n_j}(\xi)) d\xi \right\| + \int_0^t \| f_2(\xi, x_{n_k}(\xi)) - f_2(\xi, x_{n_j}(\xi))\|\, d\xi \\
&\leqslant 2\epsilon + \int_0^t u_C(\xi)\, \| x_{n_k}(\xi) - x_{n_j}(\xi)\|\, d\xi \\
&\leqslant 2\epsilon \exp\left[\int_0^t u_C(\xi)\, d\xi\right] \qquad \text{(by Gronwall lemma)},
\end{aligned}$$

so that $(x_{n_k})_k$ is a Cauchy sequence—hence convergent—in Y. From now on, we may repeat the argument well known in the finite dimensional case. We sketch it. For each n, let $T_n : Y \to Y$ be the Tonnelli approximant defined by

$$T_n(x)(t) = \begin{cases} x(t) & \text{for } t \leqslant \dfrac{b}{n}, \\ x(t) - \displaystyle\int_0^{t-b/n} f(\xi, x(\xi))\, d\xi & \text{for } t \geqslant \dfrac{b}{n}. \end{cases}$$

It is easily seen that $\lim_n T_n = T$ uniformly and that T and T_n are all continuous. By examining each interval $[ib/n, (i+1)b/n]$ $(i = 1, \ldots, n-1)$ one sees that every T_n is a bijection $Y \to Y$ and that T_n^{-1} is continuous. Hence $T^{-1}(y) \neq \varnothing$ by 1.1, while $T^{-1}(y)$ is an R_δ by 2.4 ($T^{-1}(y)$ being compact because T is a proper map). Q.E.D.

Acknowledgment

The author is grateful to G. Prodi and G. Stampacchia for valuable conversations during the preparation of the manuscript, to F. E. Browder for a preprint of [3], and to E. Michael for the reference [7, Theorem 3.4].

Added in Proofs. For some generalizations and comments, see author's paper "Applications of Topology to Analysis: On the topological properties of the set of fixed points of nonlinear operators," to appear in *Confer. Sem. Mat. Univ. Bari.*

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